MATH 4242 Quiz 1

Name:________________
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Let $V = \mathbb{R}^2$ the real vector space, and define

$$U = \{(x, y)|y = x\}$$
$$W = \{(x, y)|y = -x\}$$

(1) Prove that both U and W are subspaces of V. (4 pts)

Proof. Let u_1, u_2 be arbitrary vectors in U, then $u_1 = (a, a)$ and $u_2 = (b, b)$ for some $a, b \in \mathbb{R}$. Then $u_1 + u_2 = (a + b, a + b)$ satisfies the defining equation of U, thus is also a vector in U.

For any $c \in \mathbb{R}$, $cu_1 = (ca, ca) \in \mathbb{R}$. Therefore U is closed under addition and scalar multiplication, thus is a subspace. Proof for W is similar. □

(2) Describe their sum $U + W$. Is it a direct sum? (Provide a counterexample or a proof.) (6 pts)

Proof.

$$U + W = \{(a, a) + (b, -b)|a, b \in \mathbb{R}\} = \{(a + b, a - b)|a, b \in \mathbb{R}\}$$

We will show that this equals to $\mathbb{R}^2$. For any arbitrary vector $(c, d) \in \mathbb{R}^2$, we want to show that it can be written in the form $(a + b, a - b)$. This is equivalent to solving the equations

$$a + b = c$$
$$a - b = d$$

The only solution is $a = \frac{c+d}{2}, b = \frac{c-d}{2}$. Therefore every $(c, d) \in \mathbb{R}^2$ is an element in $U + W$. Since $U + W$ is a subspace of $\mathbb{R}^2$. Therefore $U + W = \mathbb{R}^2$.

Furthermore, because of the uniques of the above solution. Every (c, d) can be uniquely written as

$$(c, d) = (\frac{c + d}{2}, \frac{c + d}{2}) + (\frac{c - d}{2}, -\frac{c - d}{2})$$

where $(\frac{c+d}{2}, \frac{c+d}{2}) \in U$ and $(\frac{c-d}{2}, -\frac{c-d}{2}) \in W$. Thus $U + W = U \oplus W$ is a direct sum. □