Math 4242 Homework 3

(1) Let $V = \mathbb{R}^3$ and $W = \mathbb{R}_{\leq 2}[x]$. Let $T(a,b,c) = a + b(x-1) + c(x-1)^2$. Is T linear? If so, identify a basis for V and W and write down the matrix $\mathcal{M}(T)$.

Proof. T is linear because $T((a_1,b_1,c_1)+(a_2,b_2,c_2)) = (a_1+a_2)+(b_1+b_2)(x-1)+(c_1+c_2)(x-1)^2 = T(a_1,b_1,c_1)+T(a_2,b_2,c_2)$. And $T(ka,kb,kc) = k(a+b(x-1)+c(x-1)^2) = kT(a,b,c)$.

Consider e_1, e_2, e_3 the standard basis for V, and let $w_1 = 1, w_2 = (x-1), w_3 = (x-1)^2$, which form a basis for W. (Can check that they are linearly independent, and the number of vectors is equal to the dimension of W, so they must form a basis).

Now we calculate $\mathcal{M}(T)$.

$T(e_1) = 1 = w_1$, $T(e_2) = (x-1) = w_2$, and $T(e_3) = (x-1)^2 = w_3$, thus $\mathcal{M}(T) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

If we use the usual basis $1, x, x^2$ for W, then since $T(e_1) = 1$, $T(e_2) = x-1$, and $T(e_3) = x^2-2x+1$, we have that $\mathcal{M}(T) = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$

□

(2) Consider the linear map $T : M_{2,2}(\mathbb{R}) \to \mathbb{R}^2$ given by

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a-b, c+d)$$

Find a basis for $\mathrm{Ker}(T)$ and $\mathrm{Img}(T)$.

Proof. The kernal is $\ker(T) = \{\begin{bmatrix} a & a \\ c & -c \end{bmatrix} : a, c \in \mathbb{R}\}$ Note that any matrix in $\ker(T)$ can be written uniquely as

$$\begin{bmatrix} a & a \\ c & -c \end{bmatrix} = a\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}.$$

Therefore $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}$ for a basis for $\ker(T)$.

We know that $dim(\mathrm{Img}(T)) = \dim(M_{2,2}) - \dim(\ker(T)) = 4 - 2 = 2$. Note that this is the same dimension as $\mathbb{R}^2$, but $\mathrm{img}(T) \subseteq \mathbb{R}^2$, we must have that $\mathrm{Img}(T) = \mathbb{R}^2$. Thus a basis is $(1,0), (0,1)$. □

(3) Suppose $T \in \mathrm{End}(V)$ is an invertible map. Prove that if $v_1, \cdots, v_n$ is a basis, then $Tv_1, \cdots, Tv_n$ is also a basis.

Proof. Know that $\dim(V) = n$. It suffices to show that $Tv_1, \cdots, Tv_n$ is linearly independent.

Suppose $0 = a_1Tv_1 + \cdots + a_nTv_n$, we need to show that the only possibility is that $a_1 = \cdots = a_n = 0$. Left multiply both sides by T^{-1} (since T is invertible), we have

$$T^{-1}0 = T^{-1}(a_1Tv_1 + \cdots + a_nTv_n)$$

This is equivalent to

$$0 = a_1v_1 + \cdots + a_nv_n$$

Using the fact that $v_1, \cdots, v_n$ are linearly independent, we conclude that the choice of $a_1, \cdots, a_n$ is unique. This completes the proof.

□

(4) Prove that (a) $(U+W)^0 = U^0 \cap W^0$ (b) $(U \cap W)^0 = U^0 + W^0$.

Proof. (a) A linear function annihilates $U + W$ if and only if it annihilates both U and W.

(b) We first show that $(U \cap W)^0 \subset U^0 + W^0$. Take any $f \in (U \cap W)^0$, i.e. f satisfy the property $f(v) = 0$ for all $v \in U \cap W$. We want to write $f = g + h$ where $g \in U^0$ and $h \in W^0$. We can simply define $g(v) = f(v)$ for all $v \in U$ and $g(v) = 0$ otherwise, and $h(v) = f(v)$ for all $v \in W$ and $h(v) = 0$ otherwise. It can be easily checked that $f = g + h$ and $g \in U^0, h \in W^0$. Thus $(U \cap W)^0 \subset U^0 + W^0$.

We next prove the other direction. Suppose $f = g + h$ with $g \in U^0$ and $h \in W^0$, we need to show that $f \in (U \cup W)^0$. It suffices to show that $f(v) = 0$ if $v \in U \cup W$. Suppose $v \in U \cup W$, then $f(v) = g(v) + h(v) = 0 + 0 = 0$, meaning that $f \in (U \cup W)^0$. we are done. □

(5) Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(x, y, z) = (2x + 3y + 4z, 3x + 4y + 5z)$. Let e_1, e_2, e_3 denote the standard basis of $\mathbb{R}^3$ and f_1, f_2 denote the standard basis of $\mathbb{R}^2$. (a) Describe the linear functionals $T^*(f_1^*)$ and $T^*(f_2^*)$. (b) Write $T^*(f_1^*)$ and $T^*(f_2^*)$ as linear combinations of e_1^*, e_2^*, e_3^*.
(Note: There was a typo in the original problem, doing this problem in either way is fine.)

Proof. (a) $T^*(f_1^*)(x, y, z) = f_1^* T(x, y, z) = f_1 * (2x + 3y + 4z, 3x + 4y + 5z) = 2x + 3y + 4z$.
$T^*(f_2^*)(x, y, z) = f_2^* T(x, y, z) = f_2 * (2x + 3y + 4z, 3x + 4y + 5z) = 3x + 4y + 5z$.
(b) $T^*(f_1^*)(x, y, z) = 2x + 3y + 4z = 2e_1^* + 3e_2^* + 4e_3^*$
$T^*(f_2^*)(x, y, z) = 3x + 4y + 5z = 3e_1^* + 4e_2^* + 5e_3^*$
□

(6) Suppose U is a subspace of V, and $\pi : V \to V/U$ the quotient map. Consider the dual of the quotient map $\pi^* \in \mathrm{Hom}((V/U)^*, V^*)$. Show that $\mathrm{Img}(\pi^*) = U^0$ and π^* is an isomorphism $(V/U)^* \cong U^0$.

Proof. Recall that the map π is defined as $\pi(v) = v + U$, and $\pi^*(f) = f \circ \pi$ for $f : V/U \to \mathbb{F}$. Moreover $\pi^*(f)(v) = f \circ \pi(v) = f(v + U)$ which equals to 0 when $v + U = U$ i.e. $v \in U$. Thus $\mathrm{img}(\pi^*) = U^0$. Therefore π^* is surjective map from $(V/U)^*$ to U^0.

To prove it's an isomorphism, we only need that it's injective, i.e. $\ker(\pi^*) = 0$.

Let f be a linear functional on V/U. Suppose $\pi^*(f) = 0$, then $f(v+U) = 0$ for all v, which means that $f = \mathbf{0}$ (the zero vector in $(V/U)^*$), therefore $\ker(\pi^*) = \{0\}$, hence π^* is injective.

Therefore $\pi*$ is a bijective linear map from $(V/U)^*$ and U^0, thus an isomorphism.
□

(7) OS 3.1.9

Proof. Done in lecture. □

(8) OS 3.1.17

Proof.

$$\langle A, A \rangle = \mathrm{tr}(A^T A) = \sum_{i=1}^{n} (A^T A)_{ii} = \sum_{i=1}^{n} \sum_{j=1}^{m} A_{ij}^T A_{ji} = \sum_{i=1}^{m} \sum_{j=1}^{n} A_{ij}^2 \geq 0$$

and

$$\langle A, A \rangle = \sum_{i=1}^{m} \sum_{j=1}^{n} A_{ij}^2 = 0 \iff (A_{ij} = 0 \quad \forall i, j) \iff A = 0$$

Since $\mathrm{tr}(X^T) = \mathrm{tr}(X)$, $\mathrm{tr}(X + Y) = \mathrm{tr}(X) + \mathrm{tr}(Y)$, $\mathrm{tr}(\lambda X) = \lambda \mathrm{tr}(X)$ for every matrix X, Y, we have

$$\begin{aligned} \langle A, B \rangle &= \mathrm{tr}(B^T A) = \mathrm{tr}((B^T A)^T) = \mathrm{tr}(A^T B) = \langle B, A \rangle, \\ \langle \lambda A + B, C \rangle &= \mathrm{tr}(C^T (\lambda A + B)) = \mathrm{tr}(\lambda C^T A + C^T B) = \lambda \mathrm{tr}(C^T A) + \mathrm{tr}(C^T B) \\ &= \lambda \langle A, C \rangle + \langle B, C \rangle. \end{aligned}$$

□