**Math 4242 Homework 2**

(1) Let $V$ be a vector space over $\mathbb{F}$, and $v_1, \cdots, v_n \in V$. Show that $\text{span}(v_1, \cdots, v_n)$ is the smallest subspace of $V$ that contains all of $v_1, \cdots, v_n$.

*Proof.* Let $W$ be any vector space containing $v_1, \cdots, v_n$. We need to show that $\text{span}(v_1, \cdots, v_n) \subseteq W$, i.e. $v \in W$ for all $v \in \text{span}(v_1, \cdots, v_n)$. By definition of span an arbitrary element of $\text{span}(v_1, \cdots, v_n)$ looks like $v = a_1 v_1 + \cdots + a_n v_n$. Since $W$ is a subspace, it's closed under linear combination, so $v \in W$, thus $\text{span}(v_1, \cdots, v_n) \subseteq W$. □

(2) Let $V$ be a vector space, and $U_1, \cdot, U_m$ subspaces of $V$. Prove that

$$\text{span}(U_1 \cup \cdots \cup U_m) = U_1 + \cdots + U_m$$

*Proof.* Since $U_1 + \cdots + U_m$ contains $U_1 \cup \cdots \cup U_m$, by previous problem, we know that $\text{span}(U_1 \cup \cdots \cup U_m) \subseteq U_1 + \cdots + U_m$. To show they are equal, we still need that $U_1 + \cdots + U_m \subseteq \text{span}(\bigcup_{i=1}^m U_i)$.

Take any $u \in U_1 + \cdots + U_m$, it can be written as $u_1 + \cdots + u_m$ where $u_i \in U_i$. This is automatically in $\text{span}(U_1 \cup \cdots \cup U_m)$ because it's a linear combination of vectors in the union. Thus $U_1 + \cdots + U_m \subset U_1 \cup \cdots \cup U_m$, we are done. □

(3) Prove that $\text{span}(v_1, \cdots, v_n) = \text{span}(v_1) \oplus \cdots \oplus \text{span}(v_n)$ if and only if $v_1, \cdots, v_n$ are linearly independent.

*Proof.* ($\Longrightarrow$) Suppose $\text{span}(v_1, \cdots, v_n) = \text{span}(v_1) \oplus \cdots \oplus \text{span}(v_n)$. Then $0 \in \text{span}(v_1, \cdots, v_n)$ can be uniquely written as a sum $0 = w_1 + \cdots + w_n$ where $w_i \in \text{span}(v_i)$ (i.e. $w_i = k_i a_i$ for some $k_i \in \mathbb{F}$.) That is to say $0 = k_1 v_1 + \cdots k_n v_n$ for unique $k_1, \cdots, k_n$, thus by definition $v_1, \cdots, v_n$ are linearly independent.

($\Longleftarrow$) If $v_1, \cdots, v_n$ linearly independent, then any $v \in \text{span}(v_1, \cdots, v_n)$ can be uniquely written as $v = a_1 v_1 + \cdots + a_n v_n$ for some $a_1, \cdots, a_n \in \mathbb{F}$. Now since $a_i v_i \in \text{span}(v_i)$, and this is the only way to write $v$ as sum of vectors in $\text{span}(v_i)$. Hence $\text{span}(v_1) \oplus \cdots \oplus \text{span}(v_n)$ must be a direct sum. □

(4) OS 2.1.12
(5) OS 2.1.13
(6) OS 2.2.29
(7) OS 2.3.3
(8) OS 2.3.18
(9) OS 2.4.22
(10) OS 2.4.27
(11) ~~OS 3.1.9~~
(12) ~~OS 3.1.17~~

Optional (do not submit)

- 2.4.23
- 2.4.27
- 3.1.27