MATH 4242 Exam 1 Practice Problems

(1) Compute the LU factorization of the following matrix. (Hint: Use Gaussian elimination)

$$\begin{bmatrix} 1 & 5 & 0 \\ 3 & 7 & 11 \\ 0 & -4 & 2 \end{bmatrix}$$

Proof. Use Gaussian elimination to turn the matrix in to U, and L is the product of the matrices corresponding to the elementary row operations. □

(2) Let $V = \mathbb{F}_{\leq m}[x]$ be the vector space of polynomials with degree at most m.
(a) Let $U = \{f \in V | f(1) = 0\}$. Is U a subspace of V?
(b) Let $W = \{f \in V | \deg(f) \text{ is even}\}$. If W a subspace of V?

Proof. (a) If $f, g \in U$, then $(f+g)(1) = f(1) + g(1) = 0 + 0 = 0$ and $(kf)(1) = kf(1) = k \cdot 0 = 0$, thus U is closed under addition and scalar multiplication. Furthermore, the 0 polynomial satisfy the property by default. So U has a zero vector, thus is a subspace.

(b) $\deg(f)$ means the highest degree of monomials in a polynomial f. Let $f = x^2 + x + 1$ and $g = -x^2$. We have $\deg(g) = \deg(f) = 2$. But $f + g = x + 1$ and $\deg(f + g) = 1$ is not even. So W is not a subspace.

Note that if we consider even degree polynomials, which are polynomials which only contain even degree terms, then they do form a subspace. □

(3) Let $V = M_{n\times n}(\mathbb{R})$ the vector space of all $n \times n$ matrices. Let B be the set of all upper triangular matrices. Is B a subspace of V?

Proof. Yes. □

(4) Suppose $v_1, \cdots, v_4$ are some vectors in $\mathbb{R}^4$, and A is the matrix whose columns are $v_1, \cdots, v_4$. Suppose the row echelon form of A is

$$\begin{bmatrix} 3 & 1 & 7 & 1 \\ 0 & -4 & 8 & 3 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Does $v_1, \cdots, v_4$ form a basis of $\mathbb{R}^4$?
(b) Is v_3 in $\text{span}(v_1, v_2, v_4)$? If so, write v_3 as a linear combination of them.
(Hint: You may use: permuted LU factorization; some properties about matrix multiplication and invertible maps.)

Proof. (a) No, the rank of the matrix is 3 (from the number of pivots), therefore the column span has dimension 3 which is smaller than $\dim(\mathbb{R}^4) = 4$.

(b) Let $w_1, \cdots, w_4$ be the column vector of the row echelon form U of A. Use LU factorization, we know that $PA = LU$, where U is the row echelon form above. In other words, $A = P^{-1}LU$, and set $P^{-1}L = K$. We know that P and L are all products of invertible matrices so K must be also invertible. By properties of matrix product, we have that $w_i = Kv_i$. Since $w_1, \cdots, w_4$ are in row echelon form, we know that w_1, w_2, w_4 (the columns that contain a pivot) form a basis of the column space. Therefore, by invertibility of K, $v_1 = K^{-1}w_1, v_2 = K^{-1}w_2, v_4 = K^{-1}w_4$ also form a basis of the column space, thus $v_3 \in \text{span}(v_1, v_2, v_4)$.

From the row echelon form matrix, we know that $w_3 = 3w_1 - 2w_2$, thus $K^{-1}w_3 = 3K^{-1}w_1 - 2K^{-1}w_2$, thus $v_3 = 3v_1 - 2v_2$. □

(5) Let $V = \mathbb{R}^2$ with basis $v_1 = (1, 2)$ and $v_2 = (0, 1)$. Let T be the map $T(a, b) = (b, a)$ for all $a, b \in \mathbb{R}$.
(a) Show that T is linear;
(b) Find the matrix of T;
(c) Find a basis for $\ker(T)$. (Hint: use row echelon form of $\mathcal{M}(T)$.

Proof. (a) omitted (b) $T(v_1) = T(1,2) = (2,1) = 2v_1 - 3v_2$ and $T(v_2) = T(0,1) = T(1,0) = v_1 - 2v_2$. Thus $\mathcal{M}(T) = \begin{pmatrix} 2 & 1 \\ -3 & -2 \end{pmatrix}$.

(c) A row echelon form of $M(T)$ is $\begin{pmatrix} 2 & 1 \\ 0 & 7/2 \end{pmatrix}$ thus $rank(T) = 2$ and thus $\dim\ker(T) = 2 - 2 = 0$, therefore $\ker(T) = \{0\}$ □

(6) Let $V = \mathrm{End}(\mathbb{R}^2)$. Let W be the set of non-invertible linear maps in V. Show that W is not a vector space.

(Hint: give an example of two non-invertible maps that sums to an invertible map. You might try to use some theorems to do this in terms of matrices).

Proof. Consider $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Both A, B are singular (non-invertible), but $A + B = I$ is invertible.

So the set of non-invertible maps is not closed under addition. □

(7) Let $V = \mathbb{R}^2$ under the standard basis and $U = \{(x,0) : x \in \mathbb{R}\}$ a subspace of V. Describe the quotient space V/U.

Proof. Suppose $v, u \in \mathbb{R}^2$ such that $v + U = u + U$, i.e. $\{(x + v_1, v_2) | x, v_1, v_2 \in \mathbb{R}\} = \{(x + u_1, u_2) | x, u_1, u_2 \in \mathbb{R}\}$ which is true iff $v_2 = w_2$. Therefore two vectors in V belong to the same coset in V/U if they have the same second entry. Therefore the elements of V/U are parametrized by $\mathbb{R}$ according the second entry of the vectors.

In particular, $V/U = \{U(k) | k \in \mathbb{R}\} \cong \mathbb{R}$ where $U(k) := \{(x,k) | x \in \mathbb{R}\}$. □

(Hint: What is the map whose kernel is U?)

(8) Is the following matrix positive definite?

$$A = \begin{pmatrix} 1 & -1 & 3 \\ -1 & 3 & -1 \\ 3 & -1 & 12 \end{pmatrix}$$

Does $\langle x, y \rangle = x^T A y$ define an inner product on $\mathbb{R}^3$?

(9) Find the value for a, b such that the matrix A is orthogonal.

$$A = \frac{1}{3}\begin{pmatrix} 2 & -2 & b \\ 1 & 2 & 2 \\ a & 1 & -2 \end{pmatrix}$$

Proof. $a = 2, b = 3$. □