HW 4-6

♢ 3.2.7. Using (3.20), $\|\mathbf{v}-\mathbf{w}\|^2 = \langle \mathbf{v}-\mathbf{w}, \mathbf{v}-\mathbf{w}\rangle = \|\mathbf{v}\|^2 - 2\langle \mathbf{v}, \mathbf{w}\rangle + \|\mathbf{w}\|^2$
$= \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2 - 2\,\|\mathbf{v}\|\,\|\mathbf{w}\|\,\cos\theta.$

3.4.2. For instance, $q(1,0)=1$, while $q(2,-1)=-1$.

3.4.27. $K = \begin{pmatrix} 2 & 0 & \frac{2}{3} & 0 \\ 0 & \frac{2}{3} & 0 & \frac{2}{5} \\ \frac{2}{3} & 0 & \frac{2}{5} & 0 \\ 0 & \frac{2}{5} & 0 & \frac{2}{7} \end{pmatrix}$ is positive definite since $1, x, x^2, x^3$ are linearly independent.

♢ 3.4.33. Let $\langle \mathbf{x}, \mathbf{y}\rangle = \mathbf{x}^T K \mathbf{y}$ be the corresponding inner product. Then $k_{ij} = \langle \mathbf{e}_i, \mathbf{e}_j\rangle$, and hence K is the Gram matrix associated with the standard basis vectors $\mathbf{e}_1, \ldots, \mathbf{e}_n$.

3.6.37. (a) No, (b) no, ★ (c) no, (d) yes, ★ (e) yes.

♢ 8.2.10. (a) If $A\mathbf{v} = \lambda\mathbf{v}$, then $A(c\mathbf{v}) = cA\mathbf{v} = c\lambda\mathbf{v} = \lambda(c\mathbf{v})$ and so $c\mathbf{v}$ satisfies the eigenvector equation for the eigenvalue λ. Moreover, since $\mathbf{v} \neq \mathbf{0}$, also $c\mathbf{v} \neq \mathbf{0}$ for $c \neq 0$, and so $c\mathbf{v}$ is a bona fide eigenvector. (b) If $A\mathbf{v} = \lambda\mathbf{v}$, $A\mathbf{w} = \lambda\mathbf{w}$, then

$$A(c\mathbf{v} + d\mathbf{w}) = cA\mathbf{v} + dA\mathbf{w} = c\lambda\mathbf{v} + d\lambda\mathbf{w} = \lambda(c\mathbf{v} + d\mathbf{w}).$$

★ (c) Suppose $A\mathbf{v} = \lambda\mathbf{v}$, $A\mathbf{w} = \mu\mathbf{w}$. Then $\mathbf{v}$ and $\mathbf{w}$ must be linearly independent as otherwise they would be scalar multiples of each other and hence have the same eigenvalue. Thus, $A(c\mathbf{v} + d\mathbf{w}) = cA\mathbf{v} + dA\mathbf{w} = c\lambda\mathbf{v} + d\mu\mathbf{w} = \nu(c\mathbf{v} + d\mathbf{w})$ if and only if $c\lambda = c\nu$ and $d\mu = d\nu$, which, when $\lambda \neq \mu$, is only possible when either $c = 0$ or $d = 0$.

8.2.12. True — by the same computation as in Exercise 8.2.10(a), $c\mathbf{v}$ is an eigenvector for the same (real) eigenvalue λ.

♢ 8.2.32. (a) $\det(B - \lambda\,\mathrm{I}) = \det(S^{-1}AS - \lambda\,\mathrm{I}) = \det\left[S^{-1}(A - \lambda\,\mathrm{I})S\right]$
$= \det S^{-1} \det(A - \lambda\,\mathrm{I}) \det S = \det(A - \lambda\,\mathrm{I}).$

(b) The eigenvalues are the roots of the common characteristic equation. (c) Not usually. If $\mathbf{w}$ is an eigenvector of B, then $\mathbf{v} = S\mathbf{w}$ is an eigenvector of A and conversely.

★ (d) Both have 2 as a double eigenvalue. Suppose $\begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix} = S^{-1}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} S$, or, equivalently, $S\begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} S$ for some $S = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$. Then, equating entries, we must have $x - y = 2x$, $x + 3y = 0$, $z - w = 0$, $z + 3w = 2w$, which implies $x = y = z = w = 0$, and so $S = \mathrm{O}$, which is not invertible.

8.2.2. (a) The eigenvalues are $e^{\pm\,\mathrm{i}\,\theta} = \cos\theta \pm \mathrm{i}\,\sin\theta$ with eigenvectors $\begin{pmatrix} 1 \\ \mp\,\mathrm{i} \end{pmatrix}$. They are real only for $\theta = 0$ and π.

8.3.2. (a) Eigenvalue: 2; eigenvector: $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$; not complete.

(c) Eigenvalues: $1 \pm 2\,\mathrm{i}$; eigenvectors: $\begin{pmatrix} 1 \pm \mathrm{i} \\ 2 \end{pmatrix}$; complete.

(e) Eigenvalue 3 has eigenspace basis $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$; not complete.

★ (g) Eigenvalue 3 has eigenspace basis $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$; eigenvalue -2 has $\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$; not complete.

★ (i) Eigenvalue 0 has eigenspace basis $\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 0 \\ 1 \end{pmatrix}$; eigenvalue 2 has $\begin{pmatrix} -1 \\ 1 \\ -5 \\ 1 \end{pmatrix}$; not complete.

◇ 8.3.11. As in Exercise 8.2.32, if $\mathbf{v}$ is an eigenvector of A then $S^{-1}\mathbf{v}$ is an eigenvector of B. Moreover, if $\mathbf{v}_1, \ldots, \mathbf{v}_n$ form a basis, so do $S^{-1}\mathbf{v}_1, \ldots, S^{-1}\mathbf{v}_n$; see Exercise 2.4.21 for details.

◇ 8.3.21. Let S_1 be the eigenvector matrix for A and S_2 the eigenvector matrix for B. By the hypothesis $S_1^{-1} A S_1 = \Lambda = S_2^{-1} B S_2$, so $B = S_2 S_1^{-1} A S_1 S_2^{-1} = S^{-1} A S$ where $S = S_1 S_2^{-1}$.

8.4.3. (a) Real: $\{\mathbf{0}\}$, $\mathbb{R}^2$. Complex: $\{\mathbf{0}\}$, the two (complex) lines spanned by each of the complex eigenvectors $(\,\mathrm{i}, 1\,)^T, (\,-\mathrm{i}, 1\,)^T$, $\mathbb{C}^2$.

★ (c) Real: $\{\mathbf{0}\}$, the line spanned by the real eigenvector $(\,1, 0, -1\,)^T$, the plane spanned by the real and imaginary parts of the complex conjugate eigenvectors $\left(\,\frac{1}{5}, 1, -1\,\right)^T$, $(\,1, -1, 0\,)^T$, $\mathbb{R}^3$. Complex: $\{\mathbf{0}\}$, the three (complex) lines spanned by each of the complex eigenvectors $(\,1, 0, -1\,)^T, \left(\,\frac{1}{5} + \frac{3}{5}\,\mathrm{i}, 1, -1\,\right)^T, \left(\,\frac{1}{5} - \frac{3}{5}\,\mathrm{i}, 1, -1\,\right)^T$, the three (complex) planes spanned pairs of complex eigenvectors, $\mathbb{C}^3$.

8.3.16. (a) $\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \mathrm{i} & -\mathrm{i} & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathrm{i} & 0 & 0 \\ 0 & -\mathrm{i} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -\frac{\mathrm{i}}{2} & \frac{1}{2} & 0 \\ \frac{\mathrm{i}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$,

★ (b) $\begin{pmatrix} \frac{5}{13} & 0 & \frac{12}{13} \\ 0 & 1 & 0 \\ -\frac{12}{13} & 0 & \frac{5}{13} \end{pmatrix} = \begin{pmatrix} -\mathrm{i} & \mathrm{i} & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{5+12\,\mathrm{i}}{13} & 0 & 0 \\ 0 & \frac{5-12\,\mathrm{i}}{13} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{\mathrm{i}}{2} & 0 & \frac{1}{2} \\ -\frac{\mathrm{i}}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{pmatrix}$.

♡ 8.5.6. (a) If $A\mathbf{v} = \lambda\mathbf{v}$ and $\mathbf{v} \neq \mathbf{0}$ is real, then

$$\lambda\,\|\mathbf{v}\|^2 = (A\mathbf{v})\cdot\mathbf{v} = (A\mathbf{v})^T\mathbf{v} = \mathbf{v}^T A^T\mathbf{v} = -\,\mathbf{v}^T A\,\mathbf{v} = -\,\mathbf{v}\cdot(A\mathbf{v}) = -\,\lambda\,\|\mathbf{v}\|^2,$$

and hence $\lambda = 0$.

(b) Using the Hermitian dot product,

$$\lambda\,\|\mathbf{v}\|^2 = (A\mathbf{v})\cdot\overline{\mathbf{v}} = \mathbf{v}^T A^T\overline{\mathbf{v}} = -\,\mathbf{v}^T A\,\overline{\mathbf{v}} = -\,\mathbf{v}\cdot(A\mathbf{v}) = -\,\overline{\lambda}\,\|\mathbf{v}\|^2,$$

and hence $\lambda = -\,\overline{\lambda}$, so λ is purely imaginary.

(c) Since $\det A = 0$, cf. Exercise 1.9.8, at least one of the eigenvalues of A must be 0.

(d) The characteristic polynomial of $A = \begin{pmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{pmatrix}$ is $-\,\lambda^3 + \lambda(a^2+b^2+c^2)$ and hence the eigenvalues are $0, \pm\,\mathrm{i}\,\sqrt{a^2+b^2+c^2}$, and so are all zero if and only if $A = \mathrm{O}$.

(e) The eigenvalues are: (i) $\pm 2\,\mathrm{i}$, (iii) $0,\ \pm\sqrt{3}\,\mathrm{i}$.

♡ 8.5.32. (i) This follows immediately from the spectral factorization. The rows of $\Lambda\,Q^T$ are $\lambda_1\,\mathbf{u}_1^T, \ldots, \lambda_n\,\mathbf{u}_n^T$, and formula (8.37) follows from the alternative version of matrix multiplication given in Exercise 1.2.34.

(ii) (a) $\begin{pmatrix} -3 & 4 \\ 4 & 3 \end{pmatrix} = 5\begin{pmatrix} \frac15 & \frac25 \\ \frac25 & \frac45 \end{pmatrix} - 5\begin{pmatrix} \frac45 & -\frac25 \\ -\frac25 & \frac15 \end{pmatrix}.$

★ (c) $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix} = 3\begin{pmatrix} \frac16 & \frac13 & \frac16 \\ \frac13 & \frac23 & \frac13 \\ \frac16 & \frac13 & \frac16 \end{pmatrix} + \begin{pmatrix} \frac12 & 0 & -\frac12 \\ 0 & 0 & 0 \\ -\frac12 & 0 & \frac12 \end{pmatrix}.$

8.6.6. (b) Two 1×1 Jordan blocks; eigenvalues $-3, 6$; eigenvectors $\mathbf{e}_1, \mathbf{e}_2$.

★ (c) One 1×1 and one 2×2 Jordan blocks; eigenvalue 1; eigenvectors $\mathbf{e}_1, \mathbf{e}_2$.

(d) One 3×3 Jordan block; eigenvalue 0; eigenvector $\mathbf{e}_1$.

(e) One 1×1, 2×2, and 1×1 Jordan blocks; eigenvalues $4, 3, 2$; eigenvectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_4$.

8.7.2. (a) $\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} \frac{-1+\sqrt5}{\sqrt{10-2\sqrt5}} & \frac{-1-\sqrt5}{\sqrt{10+2\sqrt5}} \\ \frac{2}{\sqrt{10-2\sqrt5}} & \frac{2}{\sqrt{10+2\sqrt5}} \end{pmatrix} \begin{pmatrix} \sqrt{3+\sqrt5} & 0 \\ 0 & \sqrt{3-\sqrt5} \end{pmatrix} \begin{pmatrix} \frac{-2+\sqrt5}{\sqrt{10-4\sqrt5}} & \frac{1}{\sqrt{10-4\sqrt5}} \\ \frac{-2-\sqrt5}{\sqrt{10+4\sqrt5}} & \frac{1}{\sqrt{10+4\sqrt5}} \end{pmatrix},$

★ (b) $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$

(c) $\begin{pmatrix} 1 & -2 \\ -3 & 6 \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} \end{pmatrix} \left(5\sqrt2\right)\left(-\frac{1}{\sqrt5} \quad \frac{2}{\sqrt5}\right),$

(e) $\begin{pmatrix} 2 & 1 & 0 & -1 \\ 0 & -1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} -\frac{2}{\sqrt5} & \frac{1}{\sqrt5} \\ \frac{1}{\sqrt5} & \frac{2}{\sqrt5} \end{pmatrix} \begin{pmatrix} \sqrt7 & 0 \\ 0 & \sqrt2 \end{pmatrix} \begin{pmatrix} -\frac{4}{\sqrt{35}} & -\frac{3}{\sqrt{35}} & \frac{1}{\sqrt{35}} & \frac{3}{\sqrt{35}} \\ \frac{2}{\sqrt{10}} & -\frac{1}{\sqrt{10}} & \frac{2}{\sqrt{10}} & \frac{1}{\sqrt{10}} \end{pmatrix},$