**MATH 4242**
**Summer 2024**
**Exam 1**

Name: ______
Student ID: ______

- Exam 1 contains 7 problems. Please check to see if any page is missing.
- Time limit: July 27 10:10 am — 12:05 pm. (115 min)
- Work individually without reference to a textbook, notes, the internet, or a calculator.
- The lecture notes available from the course website is allowed. This is the only resource that is allowed during the exam. You are encouraged to refer to the theorem number in the lecture notes when you use them in your solution.
- Show your work on each problem. Specifically
  - Organize your work, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
  - Unsupported answers will not receive credit. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations will likely receive partial credit.
  - Circle your final answer for problems involving a series of computations.
  - Do NOT put answers on the back of the pages.

**Problem 1** $(3 \times 4 = 12$ pts$)$**.** Below are 5 statements about $n \times n$ invertible matrices $A$ and $B$ over $\mathbb{R}$. Circle $T$ for true and $F$ for false statements. Provide a counter-example for the false statements, and a brief explanation for the correct statements (not necessarily a rigorous proof.)

1) **T** $(AB)^{-1} = B^{-1}A^{-1}$
2) **F** $(A+B)$ must be non-singular.
   If $A$ is non-singular, then $-A$ is non-singular, but $A + (-A)$ is the zero matrix, which is singular.
3) **T** The columns of $A$ are linearly independent.
4) **F** $(\lambda A)^{-1} = \lambda(A^{-1})$.
   $(\lambda A)^{-1} = \frac{1}{\lambda}A^{-1}$

**Problem 2** $(3 \times 4 = 12$ pts$)$**.** Below are 5 statements about finite dimensional vector spaces and linear maps. Circle $T$ for true and $F$ for false statements. Provide a counter-example or a brief explanation.

1) **F** There exists subspaces $U, V$ of $\mathbb{R}^4$, such that $\dim(U) = 3$ and $\dim V = 2$ and $U \cap V = \{0\}$.
   If $U \cap V = \{0\}$, then $\dim(U+V) = \dim(U) + \dim(V) = 5$ which is larger than the dimension of $\mathbb{R}^4$. But no subspace can have larger dimension.
2) **F** Every linear map from $\mathbb{R}^4$ to $\mathbb{R}^3$ is injective.
   Every linear map from $\mathbb{R}^4$ to $\mathbb{R}^3$ is non-injective.
   For any linear map from $\mathbb{R}^4 \to \mathbb{R}^3$, $\dim(\ker(T)) = \dim(\mathbb{R}^4) - \dim(\text{Img}(T)) = 4 - \dim(\text{Img}(V))$. Since $\dim(\text{Img}(T)) \leq \dim(\mathbb{R}^3) = 3$. We have that $\dim(\ker(T)) \geq 4 - 3 = 1$. So the kernel will never be $\{0\}$, which means the map $T$ will never be injective.
3) **F** For any set of vectors $S = \{v_1, \cdots, v_m\}$ in $\mathbb{R}^n$ with $m \geq n$, there exist a subset of vectors in $S$ which form a basis of $\mathbb{R}^n$.
   This is true only when $\text{span}\{v_1, \cdots, v_m\} = \mathbb{R}^n$. For a counter example, you can take $v_1 = v_2 = \cdots = v_m$.
4) **T** Let $V_1, V_2$ be subspaces of a finite dimensional vector space $V$. If $\dim(V_1 \cap V_2) = 0$ then $V_1 \oplus V_2$ is a direct sum.

**Problem 3** $(3 \times 4 = 12$pts$)$**.** Answer the following questions with a short answer and a brief explanation.

(1) If $A$ and $B$ are matrices such that $AB = 0$, then either $A = 0$ or $B = 0$. True or False?
   False. $A = [1, 0]$ and $B = [0, 1]^T$. Then $AB$ is the zero matrix but none of them is zero.
(2) Does the zero vector belong to the span of any list of vectors?
   Yes. For any list of vectors $v_1, \cdots, v_n$, $0 = 0v_1 + \cdots + 0v_n$.
(3) Let $U$ be the set of all (real-valued) polynomials of odd degree. Is it a subspace of the vector space of all polynomials over $\mathbb{R}$?
   No. A subspace must contain the zero vector. For the polynomial vector space, the zero vector is 0. But 0 as a polynomial has degree 0 (which is even). So odd degree polynomials doesn't include zero, so they can't form a vector space.
(4) Is the set of solutions to $2x + 3y + 9 = -z$ a subspace of $\mathbb{R}^3$?
   No. $(0, 0, 9)$ is a solution, $(0, -3, 0)$ is a solution. But $(0, -3, 9)$ is not a solution.

**Problem 4** (15 pts)**.** Let $v = (a, b, c), w = (d, e, f) \in \mathbb{R}^3$ be non-zero vectors and $A = vw^t$. What's the dimension of $\ker(A)$?

$$A = vw^t = \begin{pmatrix} ad & ae & af \\ bd & be & bf \\ cd & ce & cf \end{pmatrix}$$

So the columns of $A$ are $aw, bw, cw$. Therefore by row operations we can turn $A$ in to row echelon form

$$\text{row echelon form of } A = \begin{pmatrix} ad & ae & af \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Thus $A$ has one pivots, so $rank(A) = 1$, Thus $\dim(\ker(A)) = \dim(\mathbb{R}^3) - rank(A) = 3 - 1 = 2$

**Problem 5** (15 pts)**.** Let $A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & 5 & 2 \\ 2 & -2 & 0 \end{bmatrix}$.

(a) Find an LU factorization of $A$.

(b) Solve the system of equations $Ax = b$ where $b^t = [1, 2, 1]$.

**Problem 6** (19 pts)**.** Let $T$ be a map from $\mathbb{R}^3$ to $\mathbb{R}^3$ defined by $T(a, b, c) = (a+b, a-c, b+c)$.

(a) Is $T$ linear? [3 pts]

Yeah

(b) Write down the matrix $\mathcal{M}(T)$ of $T$. [8 pts]

$T(e_1) = T(1, 0, 0) = (1, 1, 0) = e_1 + e_2$

$T(e_2) = T(0, 1, 0) = (1, 0, 1) = e_1 + e_3$

$T(e_3) = T(0, -1, 1) = -e_2 + e_3$

So $\mathcal{M}(T) = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}$

(c) Find a basis for $\ker(T)$. [8 pts]

$\ker(T) = \{(a, b, c) | a + b = a - c = b + c = 0\} = \{(a, -a, a) | a \in \mathbb{R}\}$. So $\ker(T)$ has dimension 1, with basis $(1, -1, 1)$.

**Problem 7** (15 pts)**.** Let $V = \mathbb{R}[x]$ the vector space of all real-valued polynomials (of any degree). Show that

$$\langle f, g \rangle = f(0)g(0) + \int_{-1}^{1} f'g'$$

defines an inner product on $V$.

(1) $\langle f, f \rangle = f(0)^2 + \int_{-1}^{1} (f')^2 \geq 0$. The only case when both $\langle f, f \rangle = 0$ is when $f(0) = 0$ and $f' = 0$. In this case $f(x) = 0$.

(2) $\langle f+h, g \rangle = (f(0)+h(0))g(0) + \int_{-1}^{1} (f'+h')g' = f(0)g(0) + \int_{-1}^{1} f'g' + h(0)g(0) + \int_{-1}^{1} h'g' = \langle f, g \rangle + \langle h, g \rangle$.

(3) $\langle kf, g \rangle = kf(0)g(0) + \int_{-1}^{1} kf'g' = k(f(0)g(0) + \int_{-1}^{1} f'g') = k\langle f, g \rangle$.