Super Cluster Algebras from Surfaces

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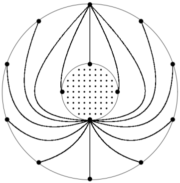
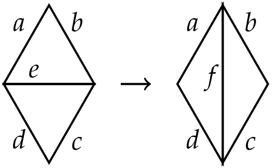
Super Cluster Algebras from Surfaces

Let F be a bordered surface with marked points on its boundary. Loosely speaking, a *cluster algebra* from F has...

1. clusters $\Longleftrightarrow$ ideal triangulations of F
2. cluster variables $\Longleftrightarrow$ "lengths" of diagonals
3. mutations $\Longleftrightarrow$ Ptolemy relations





$$ef = ab + cd$$

"Super" means *super-commutative*, i.e.

$$A = A_0 \oplus A_1 \quad \text{with relations} \quad xy = (-1)^{\bar{x}\bar{y}} yx$$

where $\bar{x} = i$ if $x \in A_i$.
More specifically, for $a, b \in A_0$ and $\theta, \sigma \in A_1$ we have

$$ab = ba \quad a\theta = \theta a \quad \theta\sigma = -\sigma\theta$$

Question

Define a super-commutative analogue of cluster algebras?

Some Super Conventions

- We will often work with a superalgebra $A = A_0 \oplus A_1$.
- Elements in A_0 are commutative, which are called *even* or *bosonic*, and will be denoted by Latin letters $x, y, z \cdots$.
- Elements in A_1 are anti-commutative, which are called *odd* or *fermionic*, and will be denoted by Greek letters $\theta, \sigma, \alpha, \beta, \cdots$.
- An element in a superalgebra has a body and a soul...

$$\underbrace{1 + x_1x_2 + x_3}_{body} + \underbrace{x_1\theta_1\theta_2 + \theta_1 + (x_1 - x_2)\theta_2}_{soul}$$

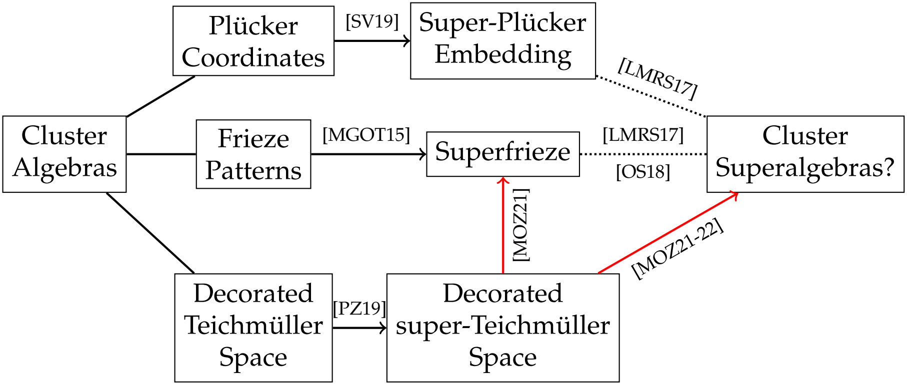
- An important fact is that odd variables square to zero: $\theta^2 = 0$.

Outline

1. **Motivation**
2. **Decorated Super Teichmüller Theory**
3. **First Formula: Super T-paths**
4. **Second Formula: Double Dimers**
5. $\mathrm{OSp}(1|2)$**-Matrix Formula**
6. **Super Fibonacci Numbers**

Outline

Motivation


Plücker Coordinates
[SV19]
Super-Plücker Embedding
[LMRS17]
Cluster Algebras
Frieze Patterns
[MGOT15]
Superfrieze
[LMRS17]
[OS18]
Cluster Superalgebras?
[MOZ21]
[MOZ21-22]
Decorated Teichmüller Space
[PZ19]
Decorated super-Teichmüller Space

A Brief History of Cluster Superalgebras

1. Ovsienko proposed an approach for cluster superalgebras [Ovs15] motivated by the study of superfriezes [MGOT15].
2. This approached was later expanded to the definition of *cluster algebras with Grassmann variables* by Ovsienko-Shapiro [OS18].
3. [LMRS17] gives a different approach, based on superfrieze patterns and $\mathrm{Gr}(2|0, 4|1)$.
4. [SV19] computed super Plüker relation for super Grassmannians and discussed certain cluster structures there-in. A more detailed discussion for the case $\mathrm{Gr}(2, 0|n, 1)$ was given in [She22] very recently.
5. In [MOZ21, MOZ22a, MOZ22b], Musiker-Ovenhouse-Z. studied the cluster structure of Penner-Zeitlin's decorated super-Teichmüller spaces.

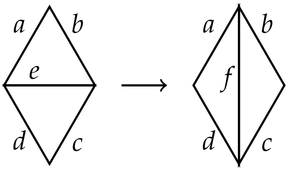
Outline

The *Teichmüller space* of a surface $F = F_g^s$ is

$$T(F) = \mathrm{Hom}\left(\pi_1(F), \mathrm{PSL}(2,\mathbb{R})\right) / \mathrm{PSL}(2,\mathbb{R}).$$

And the *decorated* Teichmüller space is the trivial $\mathbb{R}^s_{>0}$-bundle over $T(F)$, denoted $\tilde{T}(F)$. See [Pen87].

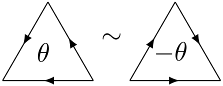
Roughly speaking, there is a λ-length associated to every pair of ideal points, satisfying the Ptolemy relation:



where $ef = ac + bd$.

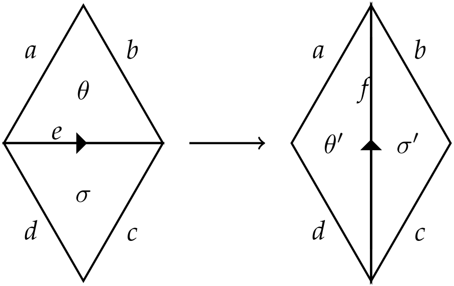
Decorated Super-Teichmüller Spaces

- By replacing $\mathrm{PSL}(2, \mathbb{R})$ with $\mathrm{OSp}(1|2)$, the super-Teichmüller space of a surface F is
$$ST(F) = \mathrm{Hom}(\pi_1(F), \mathrm{OSp}(1|2))/\,\mathrm{OSp}(1|2)$$
- In the decorated space, we have, similar to the classical case, a *super λ-length* for every pair of ideal points; and
- new coordinates called μ-invariants for every triple of ideal points (i.e. triangles).
- In addition, the super Teichmüller space consists of connected components indexed by spin structures, which are equivalence classes of orientations on the triangulations.



Super Ptolemy Relation

The Ptolemy transformation on super λ-length coordinates is given as follows.



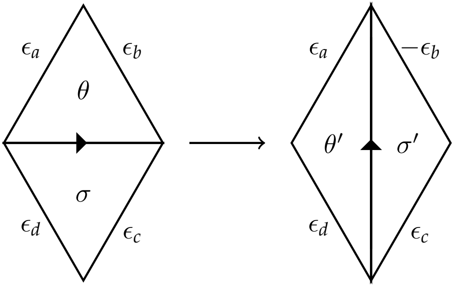
$$ef = ac + bd + \sqrt{abcd}\,\sigma\theta$$

$$\sigma' = \frac{\sigma\sqrt{bd} - \theta\sqrt{ac}}{\sqrt{ac+bd}} \quad \text{and} \quad \theta' = \frac{\theta\sqrt{bd} + \sigma\sqrt{ac}}{\sqrt{ac+bd}}$$

$$\sigma\theta = \sigma'\theta'$$

Super Ptolemy Relation

Super-flip reverse the orientation of the edge b.

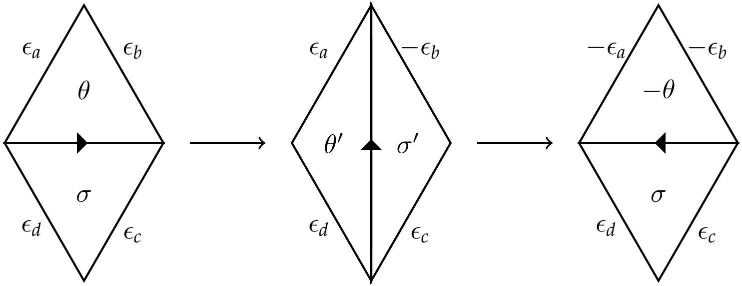


Remark

- Super Ptolemy moves are not involution: $\mu_i^8 = I$.
- The body of a super λ-length are exactly the (ordinary) λ-length in the bosonic $T(F)$.

Super Ptolemy Relation

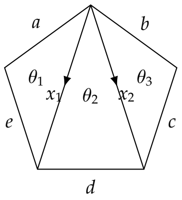
If we flip a diagonal twice:



The orientations of the triangle θ are reversed and θ is changed to $-\theta$, which corresponds to the equivalence relation mentioned before.

In other words, super Ptolemy relations are involutions only up to equivalence.

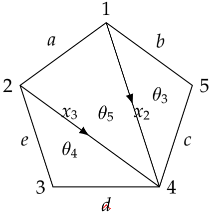
Super Ptolemy Relation - Example



Start with a Pentagon with given orientation, and we will calculate the super λ-length of the longest diagonal by flipping x_1 then x_2.

We first flip the edge x_1.

Super Ptolemy Relation - Example



After flipping x_1 to x_3, we get:

$$x_3 = \frac{ad + ex_2}{x_1} + \frac{\sqrt{adex_2}}{x_1}\theta_1\theta_2$$

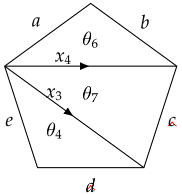
$$\theta_4 = \frac{\sqrt{ad}\,\theta_1 - \sqrt{ex_2}\,\theta_2}{\sqrt{x_1x_3}}$$

$$\theta_5 = \frac{\sqrt{ad}\,\theta_2 + \sqrt{ex_2}\,\theta_1}{\sqrt{x_1x_3}}$$

Here the red color indicates that the orientation has been reversed.

Next we flip x_2.

Super Ptolemy Relation - Example



After flipping x_2 to x_4, we have:

$$x_4 = \frac{ac + bx_3}{x_2} + \frac{\sqrt{acbx_3}}{x_2}\,\theta_5\theta_3$$

$$= \frac{acx_1 + abd + bex_2}{x_1x_2} + \frac{b\sqrt{adex_2}}{x_1x_2}\theta_1\theta_2 + \frac{\sqrt{acb\left(\frac{ad+ex_2}{x_1} + \frac{\sqrt{adex_2}}{x_1}\theta_1\theta_2\right)}}{x_2}\left(\frac{\sqrt{ad}\,\theta_2 + \sqrt{ex_2}\,\theta_1}{\sqrt{x_1x_3}}\right)\theta_3$$

$$= \frac{acx_1}{x_1x_2} + \frac{abd}{x_1x_2} + \frac{bex_2}{x_1x_2} + \frac{b\sqrt{ade}}{x_1\sqrt{x_2}}\theta_1\theta_2 + \frac{a\sqrt{bcd}}{\sqrt{x_1}x_2}\theta_2\theta_3 + \frac{\sqrt{abcd}}{\sqrt{x_1x_2}}\theta_1\theta_3$$

Question

In a cluster algebra A, any cluster variable can be expressed as a positive Laurent polynomial in the initial cluster, i.e.

$$A \subset \mathbb{R}[x_1^{\pm 1}, \cdots, x_n^{\pm 1}].$$

Questions

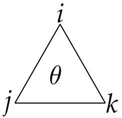
- Does the super λ-length satisfy some Laurent phenomenon?
- Is there a "positivity" for terms with anti-commuting variables?

Answers (Spoiler Alert)

- Super λ-lengths live in $\mathbb{R}[x_1^{\pm \frac{1}{2}}, \cdots, x_1^{\pm \frac{1}{2}} | \theta_1, \cdots, \theta_{n+1}]$.
- There exists an ordering on the odd variables, called *positive ordering*, such that if we multiply θ's in the positive ordering then the coefficients are positive.

Modified μ-invariants

Now we introduce some new notations to simplify the calculations.
For a triangle



Define the h-lengths

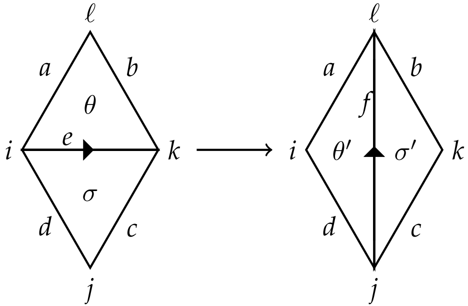
$$h^i_{jk} = \frac{\lambda_{jk}}{\lambda_{ij}\lambda_{ik}}, h^j_{ik} = \frac{\lambda_{ik}}{\lambda_{ij}\lambda_{jk}}, h^k_{ij} = \frac{\lambda_{ij}}{\lambda_{ik}\lambda_{kj}}$$

and

$$\Delta^i_{jk} := \sqrt{\frac{\lambda_{jk}}{\lambda_{ij}\lambda_{ik}}}\,\theta = \sqrt{h^i_{jk}}\,\theta, \Delta^j_{ik} := \sqrt{h^j_{ik}}\,\theta, \Delta^k_{ij} := \sqrt{h^k_{ij}}\,\theta,$$

$$\nabla^i_{jk} := \sqrt{\frac{\lambda_{ij}\lambda_{ik}}{\lambda_{jk}}}\,\theta = \sqrt{\frac{1}{h^i_{jk}}}\,\theta, \nabla^j_{ik} := \sqrt{\frac{1}{h^j_{ik}}}\,\theta, \nabla^k_{ij} := \sqrt{\frac{1}{h^k_{ij}}}\,\theta.$$

Super Ptolemy Relations Revisited

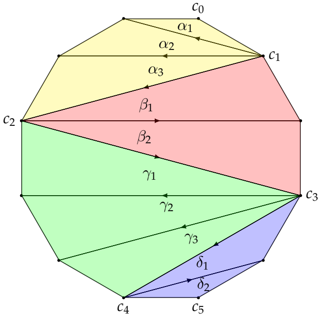


$$\lambda_{j\ell} = \frac{\lambda_{ij}\lambda_{kl} + \lambda_{i\ell}\lambda_{jk}}{\lambda_{ik}} + \nabla^{j}_{ik}\,\nabla^{\ell}_{ik}$$

$$\Delta^{k}_{j\ell} = \Delta^{k}_{ij} - \Delta^{k}_{i\ell}$$

$$\Delta^{i}_{j\ell} = \Delta^{i}_{jk} + \Delta^{i}_{k\ell}$$

Basic Constructions



From now on, only consider triangulations with a longest diagonal, and decompose into *fans* whose centers are labelled $c_1, c_2, \cdots$.

Define a *default orientation* as follows

- Edges inside each fan segments are directed away from the center.
- Others are oriented as $c_1 \to c_2 \to \cdots \to c_n$.

Define a *positive ordering* on μ-invariants.

- Going from bottom to top, append the odd variable to the left (resp. right) if the arrow is pointing left (resp. right).

$\alpha_1 > \alpha_2 > \alpha_3 > \gamma_1 > \gamma_2 > \gamma_3 > \delta_2 > \delta_1 > \beta_2 > \beta_1$

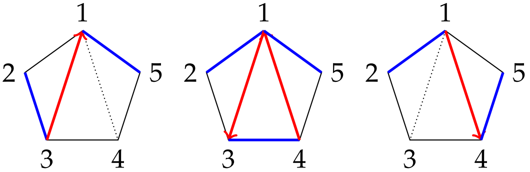
Outline

Review of Schiffler's (ordinary) T-paths

A *T-path* from i to j is a path on the triangulation T starting at vertex i, ending at j, such that

(T1) the path does not use any edge twice
(T2) the path has an odd number of edges
(T3) the even-numbered edges cross the diagonal (i, j)
(T4) The path is getting closer from i to j.

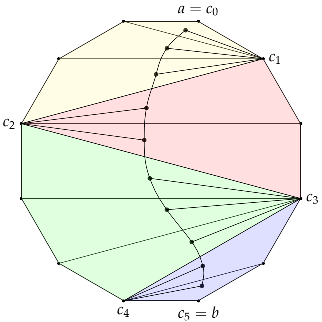
Assign a T-path a weight to $\mathrm{wt}(t) = \frac{\prod \text{odd edges}}{\prod \text{even edges}}$, then the cluster variable (λ-length) λ_{ij} is the weighted sum of all T-paths from i to j.



$\frac{x_{23}x_{15}}{x_{13}}$ $\frac{x_{12}x_{34}x_{15}}{x_{13}x_{14}}$ $\frac{x_{12}x_{45}}{x_{14}}$

$$\lambda_{25} = \sum_{t \in T_{25}} \mathrm{wt}(t) = \frac{x_{23}x_{15}}{x_{13}} + \frac{x_{12}x_{34}x_{15}}{x_{13}x_{14}} + \frac{x_{12}x_{45}}{x_{14}}$$

Super T-paths



Super T-paths are paths on the *auxiliary graph*, where all the usual T-paths moves are allowed.

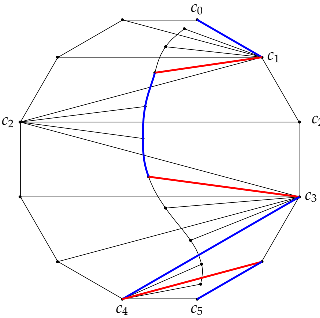
The additional moves are

- Enter or leave the internal (only) at odd steps, with $\mathrm{wt}\left(\triangle_{jk}^{i}\,\theta\right) = \Delta_{jk}^{i}$.
- Can teleport from an internal vertex to another, with weight 1.

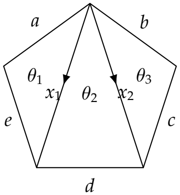
Theorem (Musiker-Ovenhouse-Z. 21)

For a default orientation, super λ-lengths are (positive) weighted sums of super T-paths, where all products of odd variables are written in the positive ordering.

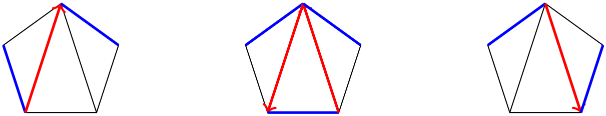
Super T-paths: Examples



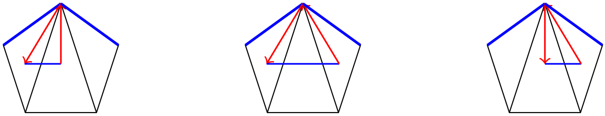
Formula for λ-lengths: Example



$\theta_1 > \theta_2 > \theta_3$



$\frac{be}{x_1}$ $\quad$ $\frac{abd}{x_1 x_2}$ $\quad$ $\frac{ac}{x_2}$

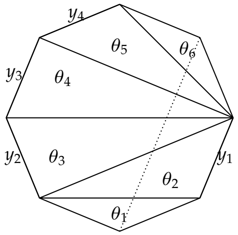


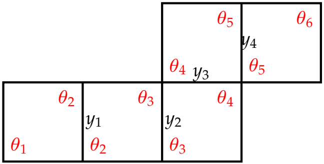
$ab\sqrt{\frac{e}{ax_1}}\theta_1\sqrt{\frac{d}{x_1x_2}}\theta_2$ $\quad$ $ab\sqrt{\frac{e}{ax_1}}\theta_1\sqrt{\frac{c}{bx_2}}\theta_3$ $\quad$ $ab\sqrt{\frac{d}{x_1x_2}}\theta_2\sqrt{\frac{c}{bx_2}}\theta_3$

Outline

Snake Graphs

Ordinary cluster variables can also be seen as perfect matchings (dimer covers) of snake graphs.

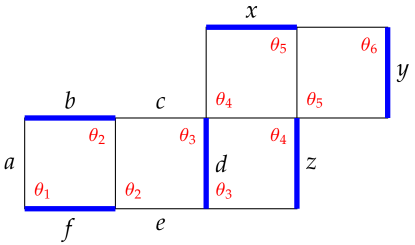




Dimer Covers on Snake Graphs

A *dimer cover* (a.k.a *perfect matching*) M of a graph G is a collection of edges such that every vertex in G is incident to exactly one edge in M.

The *weight* of a dimer cover is the product of the edge weights.



weight $= bfdxyz$

Theorem (Musiker-Schiffler, Musiker-Schiffler-Williams)

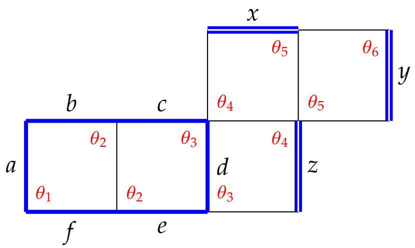
The λ-length is the given by

$$\lambda(\gamma) = \frac{1}{\text{cross}(\gamma)} \sum_{\substack{M \text{ dimer cover} \\ \text{of the snake graph}}} \text{wt}(M)$$

Double Dimer Covers

Surprisingly, the super λ-lengths naturally arise as *double dimer covers* of the same snake graph, which are unions of two dimer covers and contains single edges and doubled edges.

The weight of a double dimer cover is the product of the square root of it edges, multiplied by the odd variables on the first and last triangle of cycles.



$$\text{weight} = xyz\sqrt{abcdef}\,\theta_1\theta_3$$

Theorem (Musiker-Ovenhouse-Z. 22a)

The λ-length is the given by

$$\lambda(\gamma) = \frac{1}{\text{cross}(\gamma)} \sum_{\substack{M \text{ dimer cover} \\ \text{of the snake graph}}} \text{wt}(M)$$

SOTA

Outline

$\mathrm{OSp}(1|2)$

The orthosymplectic supergroup $\mathrm{OSp}(1|2)$ contains the set of $2|1 \times 2|1$ matrices

$$M = \left(\begin{array}{cc|c} a & b & \gamma \\ c & d & \delta \\ \hline \alpha & \beta & e \end{array}\right)$$

such that

$$e = 1 + \alpha\beta \quad e^{-1} = ad - bc \quad \alpha = c\gamma - a\delta$$
$$\beta = d\gamma - b\delta \quad \gamma = a\beta - b\alpha \quad \delta = c\beta - d\alpha$$

Note that it contains a SL_2 subgroup

$$\left(\begin{array}{cc|c} a & b & 0 \\ c & d & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$$

Special elements of $\mathrm{OSp}(1|2)$

Let x, h be even and θ odd, we define

$$E(x) = \left(\begin{array}{cc|c} 0 & -x & 0 \\ 1/x & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right) \qquad A(h|\theta) = \left(\begin{array}{cc|c} 1 & 0 & 0 \\ h & 1 & -\sqrt{h}\theta \\ \hline \sqrt{h}\theta & 0 & 1 \end{array}\right)$$

$$\rho = \left(\begin{array}{cc|c} -1 & 0 & 0 \\ 0 & -1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$$

Their inverses are given by $\rho^{-1} = \rho$, $E(x)^{-1} = \rho E(x) = E(-x)$ and

$$A(h|\theta)^{-1} = \left(\begin{array}{cc|c} 1 & 0 & 0 \\ -h & 1 & \sqrt{h}\theta \\ \hline -\sqrt{h}\theta & 0 & 1 \end{array}\right)$$

Note that $\rho A(h|\theta)\rho = A(h|-\theta)$. This corresponds to the equivalence relation of orientations in a spin structure.

We will also abbreviate

$$E_{ij} := E(\lambda_{ij}) = \left(\begin{array}{cc|c} 0 & -\lambda_{ij} & 0 \\ \lambda_{ij}^{-1} & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right) \quad A^i_{jk} := A\left(h^i_{jk} \middle| \boxed{ijk}\right) = \left(\begin{array}{cc|c} 1 & 0 & 0 \\ h^i_{jk} & 1 & -\Delta^i_{jk} \\ \hline \Delta^i_{jk} & 0 & 1 \end{array}\right)$$

A graph on T

From a triangulation T of a marked surface, we associate a graph Γ_T by putting 6 vertices inside each triangle, and connect them in the following way

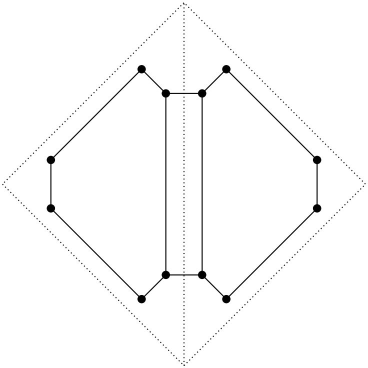


Figure: The graph Γ_T, with T in dashed lines.

Graph Connection

For a graph embedded on a surface, a *graph connection* is an assignment of a matrix to each oriented edge, such that the opposite oriented edge are assigned to its inverse.

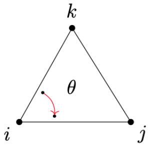
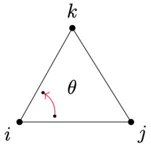
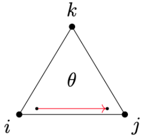
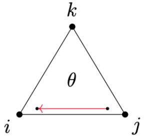
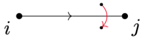
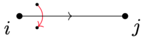
For a path in the graph, the *holonomy* is the corresponding product of matrices along the path.

If the path is a loop, then the holonomy is also called *monodromy*.

A connection is called *flat* if the monodromy around each contractible face is the identity matrix.

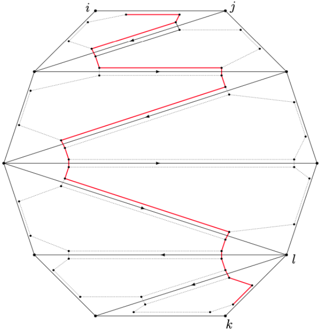
A Flat $\mathrm{OSp}(1|2)$-connection on Γ_T.

For each oriented edge of Γ_T, associate an element of $\mathrm{OSp}(1|2$ as follows.

Type (i)		${A^{i}_{jk}}^{-1}$		A^{i}_{jk}
Type (ii)		E_{ij}^{-1}		E_{ij}
Type (iii)		ρ		id

This defines a flat $\mathrm{OSp}(1|2)$-connection on Γ_T.

Matrix Formula for super λ-length



The holonomy matrix from a point near i to a point near k is given by

$$H_{ik} = \left(\begin{array}{cc|c} -\frac{\lambda_{jk}}{\lambda_{ij}} & \pm\lambda_{ik} & \nabla_{ij}^{k} \\ \pm\frac{\lambda_{jl}}{\lambda_{ij}\lambda_{kl}} & \pm\frac{\lambda_{il}}{\lambda_{kl}} & \pm\frac{1}{\lambda_{kl}}\nabla_{ij}^{l} \\ \hline \frac{1}{\lambda_{ij}}\nabla_{kl}^{j} & \pm\nabla_{kl}^{i} & 1+\star \end{array}\right)$$

In particular, the (2,2)-entry is the super λ-length up to sign.

Proof Idea

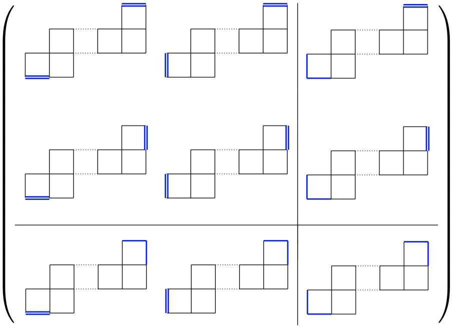
The proof uses induction in two different ways, by left-multiplication and right-multiplication.

By induction via left-multiplication, we prove the first two columns, which corresponds to flipping the diagonals from bottom to top.

By induction via right-multiplication, we prove the first two rows, which corresponds to flipping the diagonals from top to bottom.

Connection to Double Dimers

The following matrix, whose entries are weighted sum of certain double dimer covers, satisfies the $\mathrm{OSp}(1|2)$ relations.



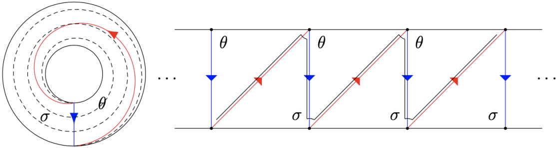
This is an analogue of 'Kuo's condensation'.

Remarks

1. The SL_2 part of our matrix formula is the same as the one given by Musiker-Williams up to signs. In particular, the usage of E and E^{-1} are swapped.
2. A similar construction for sheer coordinates of super Teichmüller spaces was given by F. Bouschbacher in his thesis. In cluster algebra language, shear coordinates are $\mathcal{X}$-type cluster variables, while λ-lengths are $\mathcal{A}$-type cluster variables.
3. The constructions given for Γ_T and the connection make sense for any triangulated surface. For a surface with non-trivial topology, the monodromy of this connection coincide with the representation $\pi_1(S) \to \mathrm{OSp}(1|2)$ described in Section 6 of Penner-Zeitlin.

Super Fibonacci Numbers

Consider an annulus with one marked point on each boundary component, and the oriented triangulation, where all λ-lengths are equal to 1.
Let z_n be λ-length of the arc connecting the two marked points which winds around the annulus $n-2$ times. This is the analogue of even indexed Fibonacci number.



In our previous paper, we showed that

$$z_n = (3 + 2\sigma\theta) z_{n-1} - z_{n-2} - \sigma\theta,$$

Super Fibonacci Numbers Continued

Let $z_n = x_{2n-5} + y_{2n-5}\sigma\theta$ and define $w_n = x_{2n-4} + y_{2n-4}\sigma\theta$, they satisfy the following recurrence.

(a) $z_n = z_{n-1} + (1+\sigma\theta)w_{n-1}$

(b) $w_n = w_{n-1} + (1+\sigma\theta)z_n - \sigma\theta$

By means of our matrix formula, we now give an interpretation for the w_n's.

$$H(z_n) = \left(\begin{array}{cc|c} -w_{n-1} & z_n & (z_n-1)\sigma + w_{n-1}\theta \\ -z_{n-1} & w_{n-1} & (z_{n-1}-1)\theta + w_{n-1}\sigma \\ \hline (z_{n-1}-1)\sigma - w_{n-1}\theta & (z_n-1)\theta - w_{n-1}\sigma & 1-(\ell_{2n-4}-2)\sigma\theta \end{array}\right)$$

where ℓ_n is the Lucas number.

Thank you for listening!

Thank You! I

- Li Li, James Mixco, B Ransingh, and Ashish K Srivastava.
  An introduction to supersymmetric cluster algebras.
  arXiv preprint arXiv:1708.03851, 2017.

- Sophie Morier-Genoud, Valentin Ovsienko, and Serge Tabachnikov.
  Introducing supersymmetric frieze patterns and linear difference operators.
  Mathematische Zeitschrift, 281(3):1061–1087, 2015.

- Gregg Musiker, Nicholas Ovenhouse, and Sylvester W Zhang.
  An expansion formula for decorated super-Teichmuüller spaces.
  Symmetry, Integrability and Geometry: Methods and Applications, 17(0):80–34, 2021.

- Gregg Musiker, Nicholas Ovenhouse, and Sylvester W Zhang.
  Double dimer covers on snake graphs from super cluster expansions.
  Journal of Algebra, 608:325–381, 2022.

Thank You! II

- Gregg Musiker, Nicholas Ovenhouse, and Sylvester W Zhang.
  Matrix formulae for decorated super Teichmüller spaces.
  arXiv preprint arXiv:2208.13664, 2022.
- Valentin Ovsienko and Michael Shapiro.
  Cluster algebras with Grassmann variables.
  arXiv preprint arXiv:1809.01860, 2018.
- Valentin Ovsienko.
  A step towards cluster superalgebras.
  arXiv preprint arXiv:1503.01894, 2015.
- Robert C Penner.
  The decorated Teichmüller space of punctured surfaces.
  Communications in Mathematical Physics, 113(2):299–339, 1987.
- Robert C Penner and Anton M Zeitlin.
  Decorated super-Teichmüller space.
  Journal of Differential Geometry, 111(3):527–566, 2019.

Thank You! III

Ralf Schiffler.
A cluster expansion formula (a_n case).
arXiv preprint math/0611956, 2006.

Ekaterina Shemyakova.
On super cluster algebras based on super Plücker and super ptolemy relations.
arXiv preprint arXiv:2206.12072, 2022.

Ekaterina Shemyakova and Theodore Voronov.
On super Plücker embedding and cluster algebras.
arXiv preprint arXiv:1906.12011, 2019.